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H-Coloring dichotomy revisited

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Abstract

The *H*-Coloring problem can be expressed as a particular case of the constraint satisfaction problem (CSP) whose computational complexity has been intensively studied under various approaches in the last several years. We show that the dichotomy theorem proved by Hell and Nešetřil [On the complexity of *H*-coloring, J. Combin. Theory Ser. B 48 (1990) 92–110] for the complexity of the *H*-Coloring problem for undirected graphs can be obtained using general methods for studying CSP, and that the criterion distinguishing the tractable cases of the *H*-Coloring problem agrees with that conjectured in [A.A. Bulatov, P.G. Jeavons, A.A. Krokhin, Constraint satisfaction problems and finite algebras, in: Proc. 27th Internat. Colloq. on Automata, Languages and Programming—ICALP'00, Lecture Notes in Computer Science, Vol. 1853, Springer, Berlin, 2000, pp. 272–282] for the complexity of the general CSP.

Keywords: Graph colouring; Complexity; Constraint satisfaction; Algebraic approach

1. Introduction

The computational complexity of the *H*-Coloring problem and related problems such as List *H*-Coloring, Counting *H*-Coloring, Restrictive *H*-Coloring has been intensively studied during the last two decades (for a comprehensive survey see [10,12]). One of the most prominent results achieved in this research direction is the *dichotomy theorem* for undirected graphs [11] that establishes that the *H*-Coloring problem is solvable in polynomial time (we shall call such problems *tractable*) if and only if *H* has a loop or is a bipartite graph; otherwise the problem is NP-complete. We call this result a dichotomy theorem, because it leaves only two possibilities for an undirected graph: to give rise either to a tractable problem or to an NP-complete problem. Notice that if $\mathrm{P} \neq \mathrm{NP}$ then there are infinitely many pairwise distinct complexity classes between P and NP [17]. In this paper, we assume $\mathrm{P} \neq \mathrm{NP}$.

The *H*-Coloring problem can be considered within a more general framework, the constraint satisfaction problem (CSP, for brevity). In the CSP associated with a finite relational structure $\mathcal{H}$ (we denote it by $\mathrm{CSP}(\mathcal{H})$), the question is whether there exists a homomorphism of a given finite relational structure to $\mathcal{H}$. Thus, the *H*-Coloring problem is a particular case of the CSP in which the involved relational structures are graphs.

One of the major research problems in studying the CSP is so-called *classification problem* aiming to distinguish those relational structures which give rise to tractable CSPs from those which do not. Several approaches to tackle the classification problem using methods from logic, algebra, game theory and database theory have been developed

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recently (see e.g. [5,7,9,15,16]), that has made it possible to achieve substantial progress [1–3,6,8,13,14,23]. This allowed Feder and Vardi [9] to conjecture that the dichotomy *tractable—NP-complete* holds for the general CSP.

The algebraic approach that has proved to be very successful uses methods and results from universal algebra, and provides a deep insight into the structure of the CSP. In particular, algebraic concepts make it possible to conjecture a plausible criterion distinguishing tractable and NP-complete CSPs [5]. (For necessary definitions and results see Section 2.) Almost all known results on the complexity of the CSP have been shown to agree with this criterion. The H-Coloring dichotomy theorem is one of the few remaining results for which it is not yet proved.

In this paper we reprove the dichotomy theorem from [11] using algebraic methods. We pursue two main goals. The first one is to illustrate how these methods can be used to obtain results about graph homomorphisms. In order to do that, in Section 2, we give an outline of the algebraic approach attempting to translate as much as possible algebraic terminology and results in graph theory terms. The second goal we achieve is to show that the criterion for the tractability of undirected H-Coloring problems is a particular case of the algebraic criterion from [5]. Theorem 1 establishes this fact. As a by-product we also get a shorter and simpler proof of the result of [11].

2. Definitions and techniques

2.1. Constraint satisfaction problem

The CSP can be equivalently defined in several ways. It is convenient for us to define the CSP as the Homomorphism problem. A *vocabulary* is a finite set of relational symbols $R_1, \ldots, R_n$ each of which has a fixed arity. A *relational structure* over the vocabulary $R_1, \ldots, R_n$ is a tuple $\mathcal{H} = (H; R_1^{\mathcal{H}}, \ldots, R_n^{\mathcal{H}})$ such that H is a non-empty set, called the *universe* of $\mathcal{H}$, and each $R_i^{\mathcal{H}}$ is a relation on H having the same arity as the symbol R_i. (We shall omit the index $\mathcal{H}$ whenever it does not lead to a confusion.) Let $\mathcal{G}, \mathcal{H}$ be relational structures over the same vocabulary $R_1, \ldots, R_n$. A *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a mapping $\varphi : G \to H$ from the universe G of $\mathcal{G}$ to the universe H of $\mathcal{H}$ such that, for every relation $R^{\mathcal{G}}$ of $\mathcal{G}$ and every tuple $(\mathbf{a}[1], \ldots, \mathbf{a}[m]) \in R^{\mathcal{G}}$, we have $(\varphi(\mathbf{a}[1]), \ldots, \varphi(\mathbf{a}[m])) \in R^{\mathcal{H}}$.

Let $\mathcal{H}$ be a relational structure over a vocabulary $R_1, \ldots, R_n$. In the *constraint satisfaction problem associated with* $\mathcal{H}$, denoted CSP($\mathcal{H}$), the question is, given a structure $\mathcal{G}$ over the same vocabulary, whether there exists a homomorphism from $\mathcal{G}$ to $\mathcal{H}$.

A (directed) graph $\mathcal{H} = (V; E)$ can be treated as a relational structure with one binary relation. Thus, the $\mathcal{H}$-Coloring problem is equivalent to CSP($\mathcal{H}$).

A relational structure $\mathcal{H}$ is said to be *tractable* if CSP($\mathcal{H}$) is tractable; it is said to be *NP-complete* if CSP($\mathcal{H}$) is NP-complete. Often it is convenient to call a set of relations Γ on H tractable if any relational structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ such that $R_1, \ldots, R_n \in \Gamma$ is tractable. The set Γ is said to be NP-complete if, for certain $R_1, \ldots, R_n \in \Gamma$, the structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ is NP-complete.

We use the standard correspondence between relations and predicates defined on the same set. In particular, we use the same symbol for a relation and for the corresponding predicate.

In [13,15], it has been shown that adding to a relational structure relations derived using certain rules does not change the complexity of the corresponding CSP. Let Γ be a set of relations. The set of relations derivable from Γ is defined to be the set of relations definable by *primitive positive formulas* (*pp-formulas* for short) involving the relations of Γ and the equality relation:

Definition 1. For any set of relations Γ over H, the set $\langle \Gamma \rangle$ consists of all relations that can be expressed using
1. relations from Γ, together with the binary equality relation on H (denoted $=_H$),
2. conjunction, and
3. existential quantification.

We say that a relation R is *definable* in a relational structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ if $R \in \langle \{R_1, \ldots, R_n\} \rangle$.

Example 1 (*Multiplication of binary relations*). Let R_1, R_2 be binary relations on a set H. Then the relation $R_1 \circ R_2$, the product of R_1, R_2, is the relation definable by the pp-formula $(R_1 \circ R_2)(x, y) = \exists z(R_1(x, z) \wedge R_2(z, y))$. We use R^n to denote the nth power of R, the relation $\underbrace{R \circ \ldots \circ R}_{n \text{ times}}$.

Example 2 (*Indicator construction Hell and Nešetřil [11]*). Let $\mathcal{I}$ be a fixed graph, and let i and j be distinct vertices of $\mathcal{I}$ such that some automorphism of $\mathcal{I}$ maps i to j and j to i. The *indicator construction* (with respect to $(\mathcal{I}, i, j)$) transforms a given graph $\mathcal{H}$ into the graph $\mathcal{H}^*$ defined to have the same vertex set as $\mathcal{H}$ and to have as the edge set the set E^* of all pairs hh' for which there is a homomorphism of $\mathcal{I}$ to $\mathcal{H}$ taking i to h and j to h'.

Let $\mathcal{I} = (W; D)$, where $W = \{i, j, i_1, \dots, i_k\}$, $\mathcal{H} = (V; E)$ and let $\mathcal{H}^* = (V; E^*)$. We treat E, E^* as binary relations on V and the elements of W as variables. It is not hard to see that E^* is definable by the following pp-formula:

$$E^*(i, j) = \exists i_1, \dots, i_k \left(\bigwedge_{xy \in D} E(x, y) \right).$$

The presence of an automorphism of $\mathcal{I}$ is equivalent to the claim that the formula is symmetric in some sense. Namely, there is a permutation of variables swopping i and j that does not change the formula.

The connection between pp-formulas and complexity is provided by the following result.

Proposition 1 (*Jeavons [13], Jeavons et al. [15]*). *Let Γ be a set of relations on a finite set. If Γ is tractable then $\langle \Gamma \rangle$ is tractable. If $\langle \Gamma \rangle$ is NP-complete then Γ is NP-complete.*

Unary definable relations (that is subsets) and definable equivalence relations play a special role in our study. Let $\mathcal{H} = (H; R_1, \dots, R_n)$ be a relational structure. Slightly abusing terminology [1] we call a *subalgebra* of $\mathcal{H}$ a unary relation definable in $\mathcal{H}$, and a *congruence* of $\mathcal{H}$ an equivalence relation definable in $\mathcal{H}$. For a subset $B \subseteq H$, the substructure of $\mathcal{H}$ *induced* by B is defined to be $\mathcal{H}_B = (B; R_1|_B, \dots, R_n|_B)$, where $R_i|_B = R_i \cap B^{m_i}$, R_i is m_i-ary. For an equivalence relation T and $a \in H$, the class of T containing a is denoted by $a/_T$ and the set of all classes of T by $H/_T$. The *quotient structure* $\mathcal{H}/_T$ is defined to be $\mathcal{H}/_T = (H/_T; R_1/_T, \dots, R_n/_T)$, where $R_i/_T = \{(a_1/_T, \dots, a_{m_i}/_T) \mid (a_1, \dots, a_{m_i}) \in R_i\}$.

Proposition 2 (*Bulatov et al. [4,5]*). *Let $\mathcal{H}$ be a relational structure, and let B and T be a subalgebra and a congruence of $\mathcal{H}$, respectively.*

(1) *If $\mathcal{H}$ is tractable then so are $\mathcal{H}_B$ and $\mathcal{H}/_T$.*
(2) *If $\mathcal{H}_B$ or $\mathcal{H}/_T$ is NP-complete then $\mathcal{H}$ is NP-complete.*

If $\mathcal{H}$ is a substructure of $\mathcal{H}'$, then a *retraction* of $\mathcal{H}'$ to $\mathcal{H}$ is a homomorphism $\varphi : \mathcal{H}' \to \mathcal{H}$ such that $\varphi(h) = h$ for all $h \in \mathcal{H}$. A structure is a *core* if it does not admit a retraction to a proper substructure. It is easy to see that every structure $\mathcal{H}'$ contains a unique, up to isomorphism, substructure $\mathcal{H}$ which is a core and admits a retraction $\varphi : \mathcal{H}' \to \mathcal{H}$; we call $\mathcal{H}$ the core of $\mathcal{H}'$. Note that if $\mathcal{H}$ is the core of $\mathcal{H}'$ then CSP($\mathcal{H}$) and CSP($\mathcal{H}'$) are polynomial time equivalent (see e.g. [15]). Therefore, we may assume that all relational structures we study are cores.

A relation of the form $C_a = \{(a)\}$, that is a unary relation containing only one tuple, is called a *constant relation*. If $\mathcal{H} = (H; R_1, \dots, R_n)$ is a relational structure then $\mathcal{H}^c$ denotes the structure $\mathcal{H}^c = (H; R_1, \dots, R_n, C_h\ (h \in H))$.

Proposition 3 (*Bulatov et al. [4,5]*). *A finite relational structure $\mathcal{H}$, which is a core, is tractable* [*NP-complete*] *if and only if $\mathcal{H}^c$ is tractable* [*NP-complete*].

For a graph $\mathcal{H}$ and a set B of vertices of $\mathcal{H}$, we use $N(B)$ to denote the union of neighbourhoods of vertices from B.

Corollary 1 (*Neibourhood*). *Let $\mathcal{H} = (V; E)$ be a graph, let $v \in V$, and let $\mathcal{H}$ be a core. If for the subgraph $\mathcal{H}' = (N(v); E')$ induced by $N(v)$ the $\mathcal{H}'$-*Coloring *problem is NP-complete, then the $\mathcal{H}$-*Coloring *problem is NP-complete.*

Proof. Since H is a core, the $\mathcal{H}$-Coloring problem is NP-complete if and only if CSP($\mathcal{H}^c$) is NP-complete. Then $N(v)$ is a subalgebra of $\mathcal{H}^c$ as the following formula shows

$$N(v)(x) = \exists y (E(x, y) \wedge C_v(y)).$$

Thus, if $\mathcal{H}'$-Coloring is NP-complete then $\mathcal{H}^c_{N(v)}$ is NP-complete and, by Proposition 2(2), so is $\mathcal{H}^c$. □

[1] In fact, unary and equivalence definable relations are subalgebras and congruences of the *universal algebra* related to $\mathcal{H}$. See [5] for details.

Corollary 2. *Let $\mathcal{H} = (V; E)$ be a graph, let $B \subseteq V$ be a subalgebra of $\mathcal{H}^c$, and let $\mathcal{H}$ be a core. If for the subgraph $\mathcal{H}' = (N(B); E')$ induced by the neighborhood of B the $\mathcal{H}'$-*Coloring *problem is NP-complete, then the $\mathcal{H}$-*Coloring *problem is NP-complete.*

Subalgebras of $\mathcal{H}^c$, where $\mathcal{H}$ is a graph, can be described in many ways. For example, they appear in [18] as *constructable sets*.

2.2. Polymorphisms

Our another main tool is polymorphisms. Every relational structure $\mathcal{H}$ has a collection of associated operations on the same universe. The unary operations associated with the structure are widely used: they are the *endomorphisms* of $\mathcal{H}$ that is homomorphisms of the structure into itself. We shall use operations of arbitrary arity.

An n-ary operation *f preserves* an m-ary relation R (or f is a *polymorphism* of R, or R is *invariant* under f) if, for any $(a_{11}, \dots, a_{m1}), \dots, (a_{1n}, \dots, a_{mn}) \in R$, the tuple $(f(a_{11}, \dots, a_{1n}), \dots, f(a_{m1}, \dots, a_{mn}))$ belongs to R. If f preserves every relation of a relational structure $\mathcal{H}$, we say that f is a polymorphism of $\mathcal{H}$. The set of all polymorphisms of $\mathcal{H}$ is denoted by $\mathsf{Pol}(\mathcal{H})$. Analogously, for a set of relations Γ, the set of all operations preserving every relation from Γ is denoted by $\mathsf{Pol}(\Gamma)$.

The *nth direct power* of a relational structure $\mathcal{H} = (H; R_1, \dots, R_k)$ is the relational structure $\mathcal{H}^n = (H^n; R_1^{\mathcal{H}^n}, \dots, R_k^{\mathcal{H}^n})$, where $((a_{11}, \dots, a_{1n}), \dots, (a_{m1}, \dots, a_{mn})) \in R_i^{\mathcal{H}^n}$ if and only if $(a_{11}, \dots, a_{m1}), \dots, (a_{1n}, \dots, a_{mn}) \in R_i$. As is easily seen, n-ary polymorphism of $\mathcal{H}$ can be viewed as a homomorphism from $\mathcal{H}^n$ to $\mathcal{H}$.

The connection between polymorphisms and definable relations is established by the following:

Proposition 4 (*see e.g. Pippenger [21], Pöschel and Kalužnin [22]*). *If Γ is a set of relations on a finite set, then* $\mathsf{Pol}(\Gamma) = \mathsf{Pol}(\langle\Gamma\rangle)$.

In particular, it follows from Proposition 4 that the subalgebras and congruences of a relational structure $\mathcal{H}$ are exactly those unary relations and equivalence relations, respectively, which are invariant under all polymorphisms of $\mathcal{H}$. Making use of Proposition 1 we infer the following connection between polymorphisms and complexity.

Corollary 3 (*Jeavons [13], Jeavons et al. [15]*). *Let $\mathcal{H}_1, \mathcal{H}_2$ be relational structures with the same universe. If* $\mathsf{Pol}(\mathcal{H}_2) \subseteq \mathsf{Pol}(\mathcal{H}_1)$ *then* $\mathrm{CSP}(\mathcal{H}_1)$ *is polynomial time reducible to* $\mathrm{CSP}(\mathcal{H}_2)$.

In a sense, Corollary 3 amounts to say that, in the study of $\mathrm{CSP}(\mathcal{H})$, a reasonable strategy is to concentrate on polymorphisms of relational structures rather than relational structures themselves. In many cases, this strategy has proved to be successful, see e.g. [1–3,5].

There are two benchmark NP-complete constraint satisfaction problems: $\mathrm{CSP}(K_n)$, $n > 2$, that is the Graph n-Colorability problem, and $\mathrm{CSP}(\mathcal{H}_{NAE})$ equivalent to the Not-All-Equal-Satisfiability problem [23]. The only relation used in the former problem is $\neq_n$, the disequality relation on an n-element set; the latter problem uses the ternary relation N on $\{0, 1\}$:

$$N = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}.$$

It is well known that the polymorphisms of these relations are so-called *essentially unary surjective operations*. An operation $f(x_1, \dots, x_n)$ on a set A is said to be essentially unary surjective if there is a bijection $g : A \to A$ and $i \in \{1, \dots, n\}$ such that $f(x_1, \dots, x_n) = g(x_i)$ for any $x_1, \dots, x_n \in A$. If every polymorphism of a graph $\mathcal{H}$ is an essentially unary surjective operation then $\mathcal{H}$ is said to be *projective* [18–20].

Proposition 5 (*Jeavons [13], Jeavons et al. [15]*). *If every polymorphism of a relational structure $\mathcal{H}$ is an essentially unary surjective operation, then $\mathcal{H}$ is NP-complete.*

In the case of 2-element structures Proposition 5 characterizes all NP-complete structures.

Proposition 6 (*Schaefer's dichotomy theorem*). *A 2-element structure $\mathcal{H}$ is tractable if and only if* $\mathsf{Pol}(\mathcal{H})$ *contains an operation which is not essentially unary surjective. In all other cases* $\mathrm{CSP}(\mathcal{H})$ *is NP-complete.*

For relational structures containing more than two elements, the sufficient condition of NP-completeness can be weakened. Let C be a set of operations on a set A, let B be a subset of A, and let T be an equivalence relation on A such that every operation from C preserves B and T. Then we denote $C|_B = \{f|_B \mid f \in C\}$, $C/_T = \{f/_T \mid f \in C\}$, where $f|_B$ denotes the restriction of f onto B, and $f/_T$ denotes the operation on $A/_T$ defined as follows: for any $a_1, \dots, a_n \in A$, $f/_T(a_1/_T, \dots, a_n/_T) = (f(a_1, \dots, a_n))/_T$. It is well known and easy to prove that if $\mathcal{H}$ is a relational structure, B is a subalgebra of $\mathcal{H}$, and T is a congruence of $\mathcal{H}$, then $(\mathsf{Pol}(\mathcal{H}))|_B \subseteq \mathsf{Pol}(\mathcal{H}_B)$ and $(\mathsf{Pol}(\mathcal{H}))/_T = \mathsf{Pol}(\mathcal{H}/_T)$.

Proposition 7 (*Bulatov et al. [4,5]*). *Let $\mathcal{H}$ be a relational structure, and let $\mathcal{H}$ be a core. Let also B be a subalgebra of $\mathcal{H}^c$, and let T be an equivalence relation on B, which is definable in $\mathcal{H}^c$. If every operation from* $((\mathsf{Pol}(\mathcal{H}^c))|_B)/_T$ *is an essentially unary surjective operation, then $\mathcal{H}$ is NP-complete.*

It is not hard to see that every polymorphism f of a relational structure of the form $\mathcal{H}^c$ satisfies the identity $f(x, \dots, x) = x$. Operations satisfying this identity are said to be *idempotent*. The property stated in Proposition 7 is the only reason known so far for the NP-completeness of a relational structure. Therefore, the following conjecture seems to be plausible [5].

Conjecture 1. *Let $\mathcal{H}$ be a relational structure, and let $\mathcal{H}$ be a core. The structure $\mathcal{H}$ is tractable if and only if, for any subalgebra B of $\mathcal{H}^c$ and any equivalence relation T on B definable in $\mathcal{H}^c$, the set* $((\mathsf{Pol}(\mathcal{H}^c))|_B)/_T$ *contains an operation which is not an essentially unary surjective operation. Otherwise it is NP-complete.*

Now we are in a position to state the main result of the paper.

Theorem 1. *For an undirected graph $\mathcal{H}$, the following conditions are equivalent:*
(a) *the $\mathcal{H}$-*Coloring *problem is tractable;*
(b) *$\mathcal{H}$ is bipartite;*
(c) *the core $\mathcal{G}$ of $\mathcal{H}$ satisfies the condition from Conjecture* 1.
*If none of the conditions holds then the $\mathcal{H}$-*Coloring *problem is NP-complete.*

2.3. Indicator and subindicator constructions vs. pp-formulas

We have already seen that the indicator construction is equivalent to a certain type of pp-formulas. In this section we show that the subindicator and edge-subindicator constructions [11] can also be represented by pp-formulas.

The subindicator construction: Let $\mathcal{J}$ be a fixed graph with specified vertices j and $k_1, k_2, \dots, k_t$. The *subindicator* construction (with respect to $\mathcal{J}, j, k_1, k_2, \dots, k_t$) transforms a given core $\mathcal{H}$ with t specified vertices $h_1, h_2, \dots, h_t$, to its subgraph $\mathcal{H}^\sim$ induced by the vertex set $V^\sim$ defined as follows: let $\mathcal{L}$ be the graph obtained from the disjoint union of $\mathcal{J}$ and $\mathcal{H}$ by identifying each k_i with the corresponding h_i $(i = 1, 2, \dots, t)$. A vertex v of $\mathcal{H}$ belongs to $V^\sim$ just if there is a retraction of $\mathcal{L}$ to H which maps the vertex j to v.

Let $\mathcal{J} = (W; D)$, where $W = \{j, k_1, k_2, \dots, k_t, v_1, \dots, v_\ell\}$, and let $\mathcal{H} = (V; E)$. Then $V^\sim$ is a subalgebra of $\mathcal{H}^c$, as the following pp-formula shows

$$V^\sim(j) = \exists k_1, \dots, k_t, v_1, \dots, v_\ell \left(\bigwedge_{xy \in D} E(x, y) \right) \wedge (C_{h_1}(k_1) \wedge \cdots \wedge C_{h_t}(k_t)).$$

The edge-subindicator construction: Let $\mathcal{J}$ be a fixed graph with specified edge jj' and t specified vertices $k_1, k_2, \dots, k_t$ such that some automorphism of $\mathcal{J}$ keeps each vertex k_i fixed while exchanging the vertices j and j'. The *edge-subindicator* construction (with respect to $\mathcal{J}, jj', k_1, k_2, \dots, k_t$) transforms a given core $\mathcal{H}$ with t specified vertices $h_1, h_2, \dots, h_t$, to its subgraph $\mathcal{H}^\wedge$ determined by those edges hh' of $\mathcal{H}$ which are images of the edge jj' under retractions of $\mathcal{L}$ (defined as above) to $\mathcal{H}$.

Let $\mathcal{J} = (W; D)$, where $W = \{j, j', k_1, k_2, \dots, k_t, v_1, \dots, v_\ell\}$, and let $\mathcal{H} = (V; E)$, $\mathcal{H}^\wedge = (V^\wedge; E^\wedge)$. Then $E^\wedge$ is a relation definable in $\mathcal{H}^c$ by the following pp-formula:

$$E^\wedge(j, j') = \exists k_1, \dots, k_t, v_1, \dots, v_\ell \left(\bigwedge_{xy \in D} E(x, y) \right) \wedge (C_{h_1}(k_1) \wedge \cdots \wedge C_{h_t}(k_t)).$$

Note that, since in both constructions the given graph $\mathcal{H}$ is a core, by Proposition 3, if the result of the transformation is NP-complete then the graph $\mathcal{H}$ is NP-complete.

3. Proof of Theorem 1

The equivalence of (a) and (b) has been proved in [11]. If (c) does not hold then, by Proposition 7, the $\mathcal{H}$-Coloring problem is NP-complete. Thus, we have to prove that (c) does not hold for any non-bipartite graph.

We take a non-bipartite graph $\mathcal{H} = (V; E)$. The graph $\mathcal{H}$ can be assumed to be the smallest one amongst all non-bipartite graphs that can be derived from $\mathcal{H}$. In particular, it is a core, has no non-bipartite subalgebras and no congruences such that the quotient graph is non-bipartite. Since $\mathcal{H}$ is a core, by Proposition 3, it is enough to show that the structure $\mathcal{H}^c$ is NP-complete. We prove that $\mathcal{H}^c$ has a subalgebra B and a congruence S of $\mathcal{H}^c_B$ such that $(\mathcal{H}_B)/_S$ is isomorphic to K_3, and hence $((\mathsf{Pol}(\mathcal{H}^c))|_B)/_S$ contains only essentially unary surjective operations.

The proof consists of two parts. In the first part we use pp-formulas to establish some useful properties of $\mathcal{H}$. This part is close to certain parts of the paper [11], but uses pp-formulas instead of subindicator constructions. In the second part, we use polymorphisms to find the required subalgebra and congruence of $\mathcal{H}^c$.

3.1. *Useful properties of the graph*

(1) $\mathcal{H}$ can be assumed to contain a triangle.

If the length of the shortest odd cycle is k, then replace $\mathcal{H}$ with $\mathcal{H}' = (V, E')$, where $E' = E^{k-2}$. Since $\mathcal{H}$ contains no cycle of length $k-2$, the graph $\mathcal{H}'$ contains no loop, and $\mathcal{H}'$ contains all the chords of the cycle of length k.

(2) Every vertex of $\mathcal{H}$ belongs to a triangle.

There is a loop at a vertex v of the graph $\mathcal{H}' = (V, E')$, where $E' = E^3$, if and only if v belongs to a triangle in $\mathcal{H}$. Therefore, every vertex of the subgraph induced by the set $E'(x, x)$ belongs to a triangle. Since $\mathcal{H}$ is minimal, $\mathcal{H} = \mathcal{H}'$.

(3) For every subalgebra B of $\mathcal{H}$, $N(B)$ is a bipartite graph. In particular, $\mathcal{H}$ does not contain K_4.

This follows from Corollary 2 and the minimality of $\mathcal{H}$.

(4) Every edge of $\mathcal{H}$ is contained in at most one triangle.

This property means that $\mathcal{H}$ avoids subgraphs shown in Fig. 1.

We consider the relation

$$R(x, y) = \exists z, t (E(x, z) \wedge E(x, t) \wedge E(y, z) \wedge E(y, t) \wedge E(z, t))$$

and its transitive closure T. The relation R consists of pairs of vertices that belong to triangles sharing an edge. So, if (4) holds then R is a subset of the equality relation. By (2), R is reflexive; therefore, $T = R^{|V|}$ is definable and is an equivalence relation. It is enough to show that the quotient graph $\mathcal{H}/_T$ contains a triangle.

To this end, we prove that T does not contain any edge of $\mathcal{H}$. Suppose for contradiction that T contains an edge. Then $\mathcal{H}$ has a homomorphic image of the graph shown in Fig. 2. Choose a, g such that the chain of rhombuses is shortest

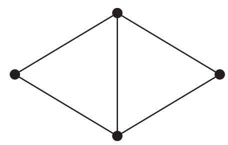

Fig. 1. A rhombus.

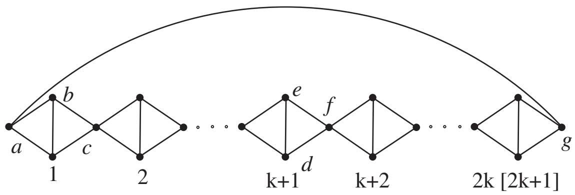

Fig. 2. A chain of rhombuses.

possible. If a, g are connected by a single rhombus, then the graph in Fig. 1 is K_4, a contradiction with (3). Suppose that the number of rhombuses is even. Then consider the set B:

$$\begin{aligned} B(x) = {} & \exists x_1, y_1, z_1, \dots, x_k, y_k, z_k (C_e(x_1) \wedge C_d(y_1) \\ & \wedge E(x_1, y_1) \wedge E(x_1, z_1) \wedge E(y_1, z_1) \wedge E(z_1, x_2) \wedge E(z_1, y_2) \\ & \wedge \cdots \wedge E(x_k, y_k) \wedge E(x_k, z_k) \wedge E(y_k, z_k) \wedge E(z_k, x)) \end{aligned}$$

(this pp-formula generalizes the subindicator construction (A5) from [11]). The variables and relations in the formula mimic the chain of rhombuses connecting e, d with g and then g with a. As is easily seen, the triangle abc belongs to B. On the other hand, if $g \in B$, then T contains an edge of $\mathcal{H}$ whose endpoints are connected with a chain of rhombuses of length $2k - 1$, a contradiction.

Finally, if the number of rhombuses is odd, then we define B through the formula

$$\begin{aligned} B(x) = {} & \exists z, x_1, y_1, z_1, \dots, x_k, y_k, z_k (C_f(z) \wedge E(z, x_1) \\ & \wedge E(z, y_1) \wedge E(x_1, y_1) \wedge E(x_1, z_1) \wedge E(y_1, z_1) \wedge E(z_1, x_2) \\ & \wedge E(z_1, y_2) \wedge \cdots \wedge E(x_k, y_k) \wedge E(x_k, z_k) \wedge E(y_k, z_k) \wedge E(z_k, x)). \end{aligned}$$

3.2. Subalgebras and congruences

In this subsection we show that there is a subalgebra B of $\mathcal{H}^c$, and a congruence S of $\mathcal{H}^c_B$ such that $(\mathcal{H}_B)/_S$ is a triangle. We prove this in a series of claims. Some of these claims are more general than we actually need.

We shall intensively use powers of triangles. Let us fix a triangle $\mathcal{T}$ with the vertex set $T = \{a, b, c\}$. Then the vertices of the graph $\mathcal{T}^k$, $k \geqslant 1$, are represented as k-tuples $x_1 \dots x_k$ of vertices of $\mathcal{T}$, and two vertices, $x_1 \dots x_k$ and $y_1 \dots y_k$, are connected if and only if x_i is connected to y_i in $\mathcal{T}$ for all $i \in \{1, \dots, k\}$; or, in other words if and only if $x_i \neq y_i$ for $i \in \{1, \dots, k\}$. Sometimes we denote elements of the $\mathcal{T}^k$ by $\overline{x} = x_1 \dots x_k$.

Claim 1. *Any two adjacent vertices in $\mathcal{T}^k$ have a unique common neighbour. Any two non-adjacent vertices in $\mathcal{T}^k$ have at least two common neighbours.*

Claim 2. *For any k, the graph $\mathcal{T}^k$ satisfies* (4), *but if an edge is added to $\mathcal{T}^k$ then the resulting graph does not satisfy* (4).

Condition (4) follows straightforwardly from Claim 1. Now, let $\overline{xy}$ be the added edge. These two vertices have two common neighbours in $\mathcal{T}^k$. The edge $\overline{xy}$ together with the two common neighbours form the graph from Fig. 1.

Recall that the *kernel* of a homomorphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is defined to be the equivalence relation on the vertex set of $\mathcal{G}_1$, such that $\ker \varphi = \{(v, w) \mid \varphi(v) = \varphi(w)\}$. Note that any congruence S of $\mathcal{G}_1$ is the kernel of some homomorphism (onto $\mathcal{G}_1/_S$, for example); but the converse is not true in general. For $k \geqslant 1$ and a set $I = \{i_1, \dots, i_\ell\} \subseteq \{1, \dots, k\}$, we denote by π_I the *projection* of $\mathcal{T}^k$ onto the set I of components, that is the homomorphism $\pi_I : \mathcal{T}^k \to \mathcal{T}^\ell$ mapping a vertex $x_1 \dots x_k$ of $\mathcal{T}^k$ to the vertex $x_{i_1} \dots x_{i_\ell}$.

Claim 3. *If a graph $\mathcal{G}$ satisfies* (4) *then, for any k and any homomorphism $\varphi : \mathcal{T}^k \to \mathcal{G}$, the following conditions hold*:
(a) *the range* $\mathrm{Im}\,(\varphi)$ *of φ is isomorphic to $\mathcal{T}^m$ for some $1 \leqslant m \leqslant k$;*
(b) $\ker \varphi = \ker \pi_I$ *for some m-element subset $I \subseteq \{1, \dots k\}$.*

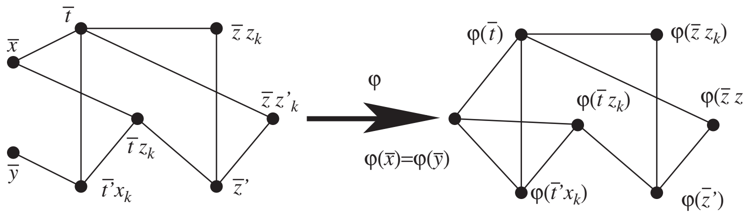

Fig. 3. Proof of Claim 3.

Let $I \subseteq \{1, \ldots, k\}$ be a maximal set such that $\ker \varphi \subseteq \ker \pi_I$. Such a set exists, because $\ker \pi_\emptyset$ is the total relation. Without loss of generality we may assume that $I = \{1, \ldots, m\}$. Thus, if $\varphi(\bar{x}) = \varphi(\bar{y})$ then $x_i = y_i$ for $i \in I$. Thus, I is the set of all coordinate positions i with this property. (Note that I can be empty.) By the maximality of I, for any $j \in \{1, \ldots, k\} - I$, $\ker \varphi \not\subseteq \ker \pi_{I \cup \{j\}}$. We prove that $\ker \pi_{\{1,\ldots,k\}-\{j\}} \subseteq \ker \varphi$.

In order to simplify the notation we assume $j = k$. Hence, there are $\bar{x}, \bar{y} \in \mathcal{T}^k$ such that $\varphi(\bar{x}) = \varphi(\bar{y})$ and $x_k \neq y_k$. Let $J = \{i \mid x_i = y_i\}$. Since $I \subseteq J$ and $k \notin J$, we may assume $J = \{1, \ldots, \ell\}$, $\ell < k$. We need to show that, for any $z_1, \ldots, z_{k-1}, z_k, z'_k \in T$, $\varphi(z_1 \ldots z_{k-1} z_k) = \varphi(z_1 \ldots z_{k-1} z'_k)$. We may assume that $z_k \neq x_k$ and $z'_k = x_k$.

By Claim 1, there is a common neighbour $\bar{t}$ of $\bar{x}$ and $\bar{z} z_k = z_1 \ldots z_{k-1} z_k$. Then let $\bar{t}' x_k = t'_1 \ldots t'_{k-1} x_k$ be a common neighbour of $\bar{t}$ and $\bar{y}$; the tuple $\bar{t} z_k = t_1 \ldots t_{k-1} z_k$ a common neighbour of $\bar{x}$ and $\bar{t}' x_k$; and $\bar{z}'$ a common neighbour of $\bar{z} z_k$, $\bar{z} z'_k = z_1 \ldots z_{k-1} z'_k$ and $\bar{t} z_k$ (see Fig. 3, left side). The images of those vertices form a subgraph shown on the right side of Fig. 3. Since G satisfies (4), we get $\varphi(\bar{t}) = \varphi(\bar{t} z_k)$, and by the same reason, $\varphi(\bar{z} z_k) = \varphi(\bar{z} z'_k)$.

Since $\ker \pi_{\{1,\ldots,k\}-\{i\}} \subseteq \ker \varphi$ for all $i \notin I$, the transitive closure of $\bigcup_{i \in \{1,\ldots,k\}-I} \ker \pi_{\{1,\ldots,k\}-\{i\}}$, i.e. $\ker \pi_I$, is a subset of $\ker \varphi$.

Claim 4. *If a graph $\mathcal{G}$ contains a triangle and satisfies* (4) *then $\mathcal{G}^c$ has a subalgebra B such that $\mathcal{G}_B$ is isomorphic to $\mathcal{T}^k$* for a certain k.

We construct a strictly increasing sequence of subgraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3 \subset \cdots$ such that $\mathcal{G}_i$ is isomorphic to $\mathcal{T}^{k_i}$ for a certain k_i. Let $\mathcal{G}_1$ be a triangle from $\mathcal{G}$, and suppose $\mathcal{G}_i$ is constructed. If G_i is a subalgebra, then we are done. Otherwise there is an (n-ary) polymorphism f of $\mathcal{G}^c$ and $v_1, \ldots, v_n \in G_i$ such that $f(v_1, \ldots, v_n) \notin G_i$. Note that f is idempotent. Let us consider f as a homomorphism from $\mathcal{G}^n$ to $\mathcal{G}$. Then its restriction onto G_i is a homomorphism from $\mathcal{G}_i^n = \mathcal{T}^{nk_i}$ to $\mathcal{G}$. By Claim 2, the image, $\mathcal{G}_{i+1}$, of f on $\mathcal{T}^{nk_i}$ is isomorphic to $\mathcal{T}^{k_{i+1}}$ for a certain k_{i+1}. By the idempotency of f, $G_i \subseteq G_{i+1}$; and by the choice of f, $G_i \neq G_{i+1}$.

Since $\mathcal{G}$ is finite, for a certain m, G_m is a subalgebra.

Therefore, $\mathcal{H}$ has a subalgebra B with this property. Let $\mathcal{H}_B$ be isomorphic to $\mathcal{T}^k$.

Claim 5. *For any (n-ary) polymorphism f of $\mathcal{T}^k$, there is a mapping $\mu : \{1, \ldots, k\} \to \{1, \ldots, n\}$ such that, for any vertices $x_1^1 \ldots x_k^1, \ldots, x_1^n \ldots x_k^n$,*

$$f\left(x_1^1 \ldots x_k^1, \ldots, x_1^n \ldots x_k^n\right) = x_1^{\mu(1)} \ldots x_k^{\mu(k)}.$$

Considering f as a homomorphism of $\mathcal{T}^{nk}$ to $\mathcal{T}^k$, by Claim 2, there is an m-element set $I \subseteq \{1, \ldots, k\} \times \{1, \ldots, n\}$ such that $\ker \varphi = \ker \pi_I$. Since f is idempotent, the range of is $\mathcal{T}^k$, and therefore $m = k$. Thus,

$$f\left(x_1^1 \ldots x_k^1, \ldots, x_1^n \ldots x_k^n\right) = \psi\left(x_{i_1}^{j_1}, \ldots, x_{i_k}^{j_k}\right),$$

where ψ is an automorphism of $\mathcal{T}^k$. The idempotency of f implies that $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$, and, since any permutation of components of tuples $x_1 \ldots x_k \in \mathcal{T}^k$ is an automorphism of $\mathcal{T}^k$, it can be assumed that $i_\ell = \ell$ for $\ell \in \{1, \ldots, k\}$. Again by the idempotency of f we get

$$x_1 \ldots x_k = f(x_1 \ldots x_k, \ldots, x_1 \ldots x_k) = \psi(x_1 \ldots x_k),$$

which means that ψ is the identity mapping.

The last claim implies that $\ker \pi_{\{1\}}$ is a congruence of the subalgebra $B = \mathcal{T}^k$. Indeed, let $f(x_1, \ldots, x_n)$ be a polymorphism of $\mathcal{H}^c$ and $x_1^1 \ldots x_k^1, \ldots, x_1^n \ldots x_k^n, y_1^1 \ldots y_k^1, \ldots, y_1^n \ldots y_k^n \in B$ such that $x_1^i = y_1^i$ for $i \in \{1, \ldots, n\}$. Then $f(x_1^1 \ldots x_k^1, \ldots, x_1^n \ldots x_k^n) = x_1^{\mu(1)} \ldots x_k^{\mu(k)}$ and $f(y_1^1 \ldots y_k^1, \ldots, y_1^n \ldots y_k^n) = y_1^{\mu(1)} \ldots y_k^{\mu(k)}$. Since $x_1^{\mu(1)} = y_1^{\mu(1)}$, f preserves $S = \ker \pi_{\{1\}}$.

Finally, as is easily seen $(\mathcal{H}_B)/_S$ is a triangle, and therefore every its polymorphism is a projection.

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